Chapter 1

Introduction

How inappropriate to call this planet Earth when it is quite clearly Ocean.
—Arthur C. Clarke

From a certain perspective in space, the Earth seems ocean entire.[1] Ocean covers 70.9% of the Earth's surface despite billions of years of continental accumulation. In epochs past, there was only ocean (Ward & Brownlee, 2000).

The ocean's part in climate and life on Earth surpasses its size. More than 90% of the heat energy added to the Earth system between 1955 and 2010 is stored in the ocean. This massive amount of energy corresponds to 36°C of atmospheric warming (Levitus *et al.*, 2012). Whatever the concerns of mankind, the increase in land surface temperature known as 'global warming' is a minor correction to the changes recorded in our warming ocean.

The many oceanic roles in climate emerge from its kaleidoscopic patchwork of motion: the froth of white-capping sea and swell, storm-like eddies spinning off the Gulf Stream, and lumbering internal waves tens to hundreds of meters tall. The ocean's rotating and density-stratified dynamics entangle each piece spanning from the planetary to the planktonic, placing detailed predictions of ocean dynamics far beyond reach of current technology. A necessary step toward forecasting climate change is thus the development of new models for ocean physics that are efficient and approximate yet still physically-based and reliable.

This dissertation contributes to that effort by seeking a deeper understanding of part of the patchwork: the interweaving of two oceanic motions called 'internal waves' and 'quasi-geostrophic flow' with spatial scales of tens to hundreds of kilometers. The methods of this dissertation are theoretical, consisting mainly of the development of models that isolate the physics of waves and flow and augmented by a small number of analytical and numerical examples. It is hoped that further analysis of the models developed in this dissertation will prove useful in interpreting both observations and numerical simulations and in developing ever-better models for oceanic circulation and the evolution of Earth's climate.

[1] http://eoimages.gsfc.nasa.gov/images/imagerecords/46000/46209/earth_pacific_lrg.jpg

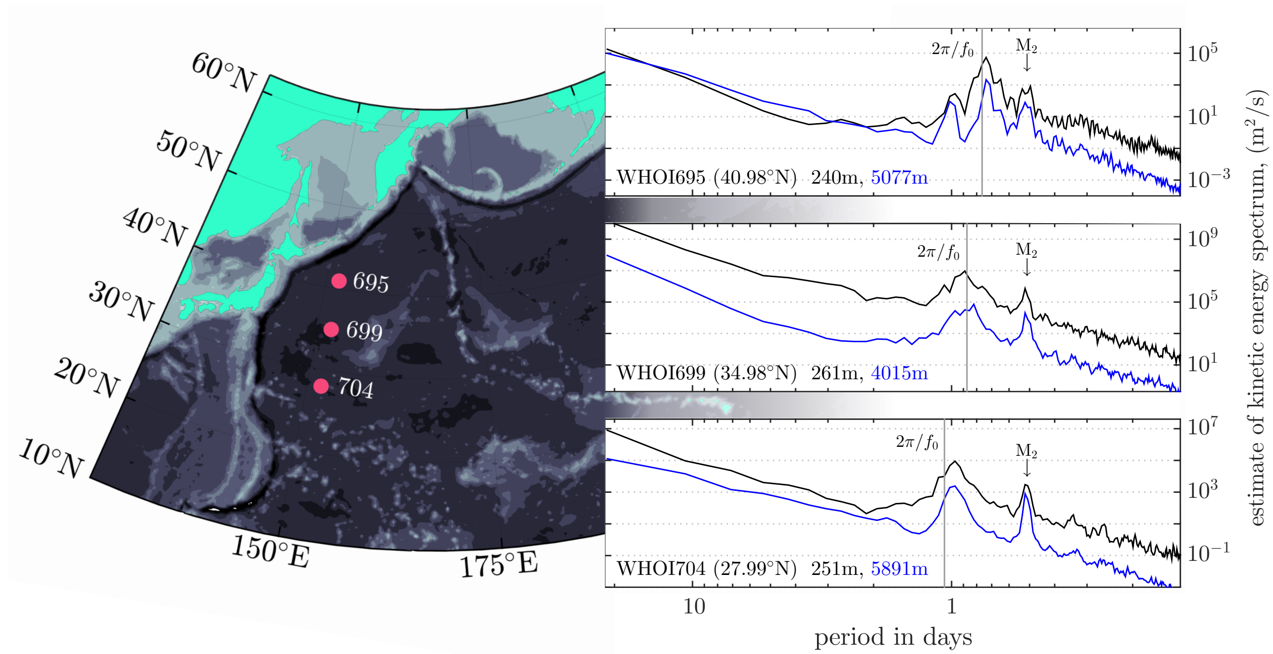

Figure 1.1: Estimates of kinetic energy frequency spectra in three one-year mooring records from the western Pacific locations shown on the map at left. At right are kinetic energy spectra from upper-ocean and abyssal instruments on each mooring. Spectral estimates are the ensemble average of spectra from 35 overlapping and Hamming-windowed 20-day segments extracted from each year-long record. The arrow and label 'M_2' marks the 12.421-hour period of the diurnal tide and a grey line indicates the $2\pi/f_0 = (2\sin\phi)^{-1}$-day inertial period at latitude ϕ. Small peaks are discernible at the mixed-harmonic period $2\pi/(f_0 + M_2) = 0.31$ and 0.34 days in data from 40.98°N and 27.99°N, respectively. WESTPAC data from OSU's Deep Water Archive[2] was provided in convenient form by Harper Simmons.

1.1 Waves and flow

Outside surface and bottom boundary layers, oceanic motion is mostly a mixture of internal waves and quasi-geostrophic flow. Waves and flow have similar horizontal space-scales of tens to hundreds of kilometers, but widely disparate time-scales ranging from a few minutes for the fastest waves to months or years for the most slowly-evolving flows. These pithy oceanic facts are evidenced by six estimates of kinetic energy frequency spectra shown in figure 1.1. The estimates are made from hourly, year-long observations of horizontal velocity in the western Pacific made during the WESTPAC experiment between the summers of 1980 and 1981[2].

Notice first the two conspicuous peaks that appear in every record: one broad and shifting with periods close to one day, and another narrow and fixed at a period of 12.421 hours. The first peak is the fingerprint of 'near-inertial waves' close to the local inertial frequency $f_0 = 4\pi\sin\phi/\text{day}$ at latitude ϕ forced by diverse mechanisms like winds and flow-bathymetry interaction. The second peak corresponds to a mix of surface tides and internal waves or 'internal tides' forced with astronomical precision by the 12.421-hour lunar semidiurnal tide. A third peak manifests at the solar and lunar diurnal periods close to one day in the record from 40.98°N which may correspond to the depth-indepedent surface tide or to the tidally-forced evanescent internal waves explored by Musgrave *et al.* (2016). Observe the logarithmic scale: the energy density at inertial and tidal peaks is 100× greater than at surrounding frequencies.

[2]http://www.cmrecords.net/quick/pacific/wp/wp.htm

Their inclination to break and churn the ocean with small-scale turbulence suggests that internal waves make an important contribution to the vertical, diapycnal mixing that sets the ocean's density stratification and draws heat and carbon into the abyss.

The inertial and tidal peaks both correspond to relatively high-frequency internal waves. Moving left from the inertial peak toward lower frequencies and longer periods, kinetic energy density first decreases to a minimum and thereafter increases to what is typically a maximum for each spectrum at the longest observed period. The sluggish, energy-containing motions associated with this leftward maximum are quasi-geostrophic flows: planetary Rossby waves, meandering currents, and slowly-spinning eddies. These flows are 'quasi-geostrophic' because their leisurely evolution over many inertial periods implies they adhere to a linear geostrophic balance between the inertial Coriolis force and pressure gradient force. Quasi-geostrophic eddies and currents contain most of the ocean's kinetic energy away from storm-whipped surface layers, and rapidly stir oceanic heat and carbon over decadal time-scales on surfaces of constant density connected to the atmosphere.

In consequence, predicting the Earth system's short-term response to rapid changes in CO_2 concentration, for example, requires an approximate description of the quasi-geostrophic stirring not explicitly resolved in coarse resolution models (Danabasoglu *et al.*, 2012; Danabasoglu & Marshall, 2007). And efforts for predicting climate evolution over long, hundred-year time-scales requires knowledge of the changing magnitude and spatial distribution of wave-driven diapycnal mixing to accurately describe abyssal absorption of carbon and slow changes in the ocean's density stratification so critical to ocean dynamics. Approximations of diapycnal mixing may require distinct components to account separately for the mixing driven by internal tides (Melet *et al.*, 2013; Green & Nycander, 2013; Olbers & Eden, 2013) and near-inertial waves (Melet *et al.*, 2014; Jochum *et al.*, 2013). A strong physical basis is necessary for such approximate descriptions of waves and flow to withstand changing atmospheric and oceanic conditions over the course of decades and centuries.

Spurred by the need to better understand internal waves and quasi-geostrophic flow and sustained by a conviction that new mathematical models can yield substantial physical intuition, this dissertation develops models that isolate the nonlinear interaction of oceanic internal waves and quasi-geostrophic flow. We focus first on evolution of wave-averaged quasi-geostrophic flow in arbitrary and prescribed field of hydrostatic internal waves chapter 2. Next, we develop two models that couple quasi-geostrophic flow to near-inertial waves and their second harmonic in chapter 3 and isolate the slow evolution of internal tides in quasi-geostrophic flow in chapter 4.

1.2 Mathematical overtures

The shape of typical frequency spectra speaks to a dichotomy among energy-containing oceanic motions. The energy-density minimum or 'spectral gap' between the conspicuous high-frequency internal wave peaks and leftward-increasing ramp of low-frequency quasi-geostrophic flow is intrinsic to the ocean's density-stratified and rotating physics: both waves and flow are fundamentally *small-amplitude* motions, or slight perturbations to the ocean's basic state of rapid rotation and strong density stratification.

The root of this oceanic dichotomy is exposed by a review of the small-amplitude, linear solutions to this dissertation's standard model for oceanic motion, the inviscid, rotating Boussinesq equations on the β-plane. The linear solutions to the rotating Boussinesq equations form the basis for the reduced models developed in 2, 3, and 4. The trek through linear landscapes ends with a glimpse into nonlinear wilds that primes needed mathematical machinery and evokes essential physical ideas.

1.2.1 Dynamics of rotating Boussinesq fluids

The rotating Boussinseq equations are posed in a reference frame that rotates with the Earth at frequency $\Omega = 2\pi/\text{day}$ and expanded around a static, background density stratification. Fluid density is decomposed into

$$\rho(\boldsymbol{x}, t) = \rho_0 + \rho_*(z) + \rho'(\boldsymbol{x}, t)\,, \tag{1.1}$$

where t is time and $\boldsymbol{x} = (x, y, z)$ are Cartesian east, north, and vertical coordinates. In (1.1), ρ_0 is an average or reference density, $\rho_*(z)$ is the background density stratification, and ρ' is the dynamic perturbation associated with fluid motion. We define the background buoyancy profile B_* and 'buoyancy' b associated with the dynamic density perturbation ρ',

$$B_*(z) \stackrel{\text{def}}{=} -\frac{g\rho_*(z)}{\rho_0} \qquad \text{and} \qquad b \stackrel{\text{def}}{=} -\frac{g\rho'}{\rho_0}\,. \tag{1.2}$$

The buoyancy b is an acceleration imposed on the fluid by deviations in density from the background profile. We also decompose pressure into hydrostatic and dynamic components. The fluid's total pressure field is decomposed into

$$-\rho_0 g z + \rho_0 P_*(z) + \rho_0 p(\boldsymbol{x}, t)\,, \tag{1.3}$$

where $P_{*z} = -g\rho_*/\rho_0$ so that $-\rho_0 g z + \rho_0 P_*$ is the hydrostatic part of pressure and $\rho_0 p$ is the dynamic part of pressure associated with fluid motion.

Two important frequencies intrinsic to density stratification and rotation are the buoyancy frequency, N, and inertial or Coriolis frequency, f. The buoyancy frequency is

$$N^2 \stackrel{\text{def}}{=} \frac{\mathrm{d}B_*}{\mathrm{d}z} = -\frac{g}{\rho_0}\frac{\mathrm{d}\rho_*}{\mathrm{d}z}\,. \tag{1.4}$$

N is the frequency of gravity- or buoyancy-driven oscillations induced by small *vertical* displacements of fluid. The inertial frequency is

$$f \stackrel{\text{def}}{=} 2\Omega \sin\phi\,, \tag{1.5}$$
$$\approx f_0 + \beta y\,, \tag{1.6}$$

where ϕ is latitude. In (1.6) we move into a Cartesian reference frame which is tangent to the Earth's surface at the reference latitude ϕ_0 and make the 'β-plane approximation'. On the β-plane, f is expanded around ϕ_0 so that the local inertial frequency is $f_0 = 2\Omega \sin\phi_0$ and the latitudinal variation of f is modeled by $\beta y = (2\Omega\cos\phi_0/R)\,y$, where R is the radius of the Earth. The local inertial frequency f_0 is the frequency of oscillations induced by small *horizontal* displacements of fluid and restored by the displacement's inertial advection of the background rotating velocity field.

The equations used in this dissertation follow from four crucial assumptions: *(i)* the dynamics are inviscid with negligible molecular diffusion and dissipation; *(ii)* density depends linearly on the concentration of one or more scalar quantities; *(iii)* the Boussinesq approximation is valid because density fluctuations are relatively small so that $\rho_* + \rho' \ll \rho_0$; and *(iv)* we can neglect the inertial term $2\Omega\cos\phi\,(w\hat{\boldsymbol{x}} - u\hat{\boldsymbol{y}})$ from the momentum balance because the aspect ratio H/L of considered motions is small so that $w \ll (u, v)$, where $\boldsymbol{u} = (u, v, w)$ is the fluid velocity. Note that we hold off on assuming hydrostatic balance $p = b_z$ in the vertical momentum equation until chapter 1.2.3, despite that disregarding $(2\Omega\cos\phi)\,u$ while assuming $H/L \ll 1$ and $u \gg w$ requires it. This minor slight-of-hand permits a fuller discussion of linear physics than would be possible under the hydrostatic approximation. With this caveat, the preceding definitions and assumptions lead to the rotating Boussinesq equations on the β-plane,

$$\mathrm{D}_t u - fv + p_x = 0\,, \tag{1.7}$$

$$\mathrm{D}_t v + fu + p_y = 0\,, \tag{1.8}$$

$$\mathrm{D}_t w + p_z = b\,, \tag{1.9}$$

$$\mathrm{D}_t b + wN^2 = 0\,, \tag{1.10}$$

$$u_x + v_y + w_z = 0\,. \tag{1.11}$$

where subscripts with respect to (x, y, z) or t denote partial derivatives, and D_t is the material derivative following the fluid,

$$\mathrm{D}_t \stackrel{\text{def}}{=} \partial_t + \boldsymbol{u}\cdot\boldsymbol{\nabla}\,. \tag{1.12}$$

In appendix A we show how (1.7) through (1.11) can be written in the different and useful 'wave operator form'. The Ertel potential vorticity is

$$\Pi \stackrel{\text{def}}{=} \boldsymbol{\omega}_a\cdot\boldsymbol{\nabla}\mathcal{B}\,, \tag{1.13}$$

$$\stackrel{\text{def}}{=} (f\,\hat{\boldsymbol{z}} + \boldsymbol{\omega})\cdot\left(N^2\,\hat{\boldsymbol{z}} + \boldsymbol{\nabla}b\right)\,, \tag{1.14}$$

$$= fN^2 + N^2\omega + fb_z + \boldsymbol{\omega}\cdot\boldsymbol{\nabla}b\,, \tag{1.15}$$

where $\boldsymbol{\omega}_a$ is absolute vorticity, $\mathcal{B} = B_* + b$ is the total buoyancy field, and $\boldsymbol{\omega} \stackrel{\text{def}}{=} \boldsymbol{\nabla}\times\boldsymbol{u}$ is relative vorticity with vertical component $\omega = \hat{\boldsymbol{z}}\cdot\boldsymbol{\omega} = v_x - u_y$. A remarkable property of equations (1.7) through (1.11) is the material conservation of Π, so that

$$\mathrm{D}_t\Pi = 0\,. \tag{1.16}$$

The conservation of Π expressed by (1.16) is a statement of angular momentum conservation for an effectively constant-density fluid that rotates locally with an effective angular velocity of $\boldsymbol{\omega}_a/2$ and whose extension along the axes of rotation is tracked by $\boldsymbol{\nabla}\mathcal{B}$. In other words, pulling fluid surfaces apart decreases $\boldsymbol{\nabla}\mathcal{B}$ and spins up the fluid by increasing $\boldsymbol{\omega}_a$. For the small-amplitude motion of waves and flow, fN^2 in (1.15) is by far the largest component of Π.

1.2.2 Lessons of linear dynamics

The formulation of (1.7) through (1.11) means the velocity $\boldsymbol{u}$ and buoyancy b are *departures* from a stable basic state in solid body rotation around the z-axis with angular velocity $f/2$ and density profile $\rho_0 + \rho_*$. Waves and flow are both small perturbations to this basic state with small $\boldsymbol{u}$ and b, which means they are well described by the linear terms in equations (1.7) through (1.11) obtained by assuming $\mathrm{D}_t \approx \partial_t$,

$$u_t - f_0 v + p_x = 0\,, \tag{1.17}$$
$$v_t + f_0 u + p_y = 0\,, \tag{1.18}$$
$$w_t - b + p_z = 0\,, \tag{1.19}$$
$$b_t + wN^2 = 0\,, \tag{1.20}$$
$$u_x + v_y + w_z = 0\,. \tag{1.21}$$

Equations (1.17) through (1.21) are the linearized Boussinesq equations. Their unsteady solutions are internal waves and their steady solutions are geostrophic flows.

A conservation law follows by forming $\partial_x(1.18) - \partial_y(1.17)$ and using (1.21) and (1.20),

$$\partial_t \left[v_x - u_y + \partial_z \left(\frac{f_0 b}{N^2} \right) \right] \stackrel{\text{def}}{=} N^2 Q_t = 0\,, \tag{1.22}$$

where we recall that $\omega = \hat{\boldsymbol{z}} \boldsymbol{\cdot} \boldsymbol{\omega} = v_x - u_y$ is the vertical component of vorticity. In equation (1.22) we have defined Q, the linear 'Available Potential Vorticity', or APV. Linear APV is synonymous with the standard expression for quasi-geostrophic potential vorticity. The linearized APV does not evolve in (1.17) through (1.21): for internal waves $Q = 0$ and for geostrophic flow $Q = Q(\boldsymbol{x})$ is constant in time. The general definition of nonlinear APV in chapter 2.2 is one of the main accomplishments of this dissertation. Notice that (1.22) is not equal to the linear parts of Ertel PV in (1.14). Thus internal waves generate non-trivial signatures in Π even while $Q = 0$. This point is central to the utility of APV.

Waves

When $f = f_0$ is constant, a short series of manipulations on (1.17) through (1.21) discussed in detail in appendix A leads to a single equation for w,

$$\left[\partial_t^2 \left(\triangle + \partial_z^2 \right) + f_0^2 \partial_z^2 + N^2 \triangle \right] w = 0\,, \tag{1.23}$$

where we define the horizontal Laplacian $\triangle \stackrel{\text{def}}{=} \partial_x^2 + \partial_y^2$. Equation (1.23) is the internal wave equation. When f and N are constant and the considered domain is either infinite or a periodic box, we can decompose w into the sinuosoids $w = \exp\left(\mathrm{i}\boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x} - \mathrm{i}\sigma t\right) \hat{w}(\boldsymbol{k}, \sigma)$, where σ is frequency and $\boldsymbol{k} = (k, \ell, m)$ is wavenumber. Then (1.23) implies that $\boldsymbol{k}$ and σ satisfy the *dispersion relation*,

$$\sigma^2 = \frac{f_0^2 m^2 + N^2 \left(k^2 + \ell^2\right)}{k^2 + \ell^2 + m^2}\,. \tag{1.24}$$

Equation (1.24) shows that the frequency of linear, freely-propagating internal waves always lies between f_0 and N, whether $f_0 < N$ or $N < f_0$. When N is not constant but varies slowly compared to $1/m$, equation (1.24) becomes a local approximation. A stationary phase analysis developed by Lighthill (2001) in chapters 3.7 and 3.8 of his book shows that energy in the linear, Fourier-decomposed wave field travels at the 'group velocity' $\mathbf{U} = \boldsymbol{\nabla}_{\boldsymbol{k}}\sigma$ corresponding to the vector $\boldsymbol{x}/t$ at which the phase function $\theta = \boldsymbol{k}\boldsymbol{\cdot}\boldsymbol{x}/t - \sigma$ is stationary. This indicates the group velocity of waves near frequency f_0 or N is small where σ changes slowly with $\boldsymbol{k}$.

The dispersion relation in (1.24) implies that waves with frequency close to f_0 have $(Nk/f_0 m)^2 \ll$

1 and thus large horizontal scales and small vertical scales under typical oceanic conditions where $f_0 \ll N$. These nearly-horizontally-uniform 'near-inertial' motions have small horizontal pressure gradients, so that (1.17) and (1.18) combine into

$$\mathcal{U}_t + \mathrm{i} f_0 \mathcal{U} \approx 0 \,, \qquad \text{where} \qquad \mathcal{U} \stackrel{\text{def}}{=} u + \mathrm{i} v \,. \tag{1.25}$$

The solution to (1.25) is $\mathcal{U} \approx \mathrm{e}^{-\mathrm{i} f_0 t} A(\boldsymbol{x}, t)$, where A is a near-arbitrary function of space that evolves slowly in the linear equations to reflect slight departures of $\mathcal{U}$ from the inertial frequency. When $A = A(\boldsymbol{x})$ is stationary this type of motion is often called an 'inertial oscillation', though a better name is 'pure inertial wave'. At the other end of the spectrum are motions with small horizontal scales and large vertical scales. These near-buoyancy waves have small vertical pressure gradients so that (1.19) and (1.20) merge into

$$\mathcal{W}_t + \mathrm{i} N \mathcal{W} \approx 0 \,, \qquad \text{where} \qquad \mathcal{W} \stackrel{\text{def}}{=} w + \mathrm{i} b / N \,. \tag{1.26}$$

The solution to (1.26) is $\mathcal{W} \approx \mathrm{e}^{-\mathrm{i} N t} A(\boldsymbol{x}, t)$, where again A is an near-arbitrary function of space and slowly evolves in time. In the real and heterogeneous ocean, pure inertial or buoyancy waves cannot exist. Motions are always *near*-inertial or *near*-buoyancy.

The fact that $\mathcal{U}$ and $\mathcal{W}$ have arbitrary spatial structure in (1.25) and (1.26) reflects the important fact that dispersion only weakly constrains the spatial structure of malleable near-inertial and near-buoyancy waves. The weak dispersion and correspondingly slow propagation of near-inertial and near-buoyancy waves means that oceanic heterogeneities not accounted for in the linear equations, like quasi-geostrophic flow, small-scale turbulence, or surface waves, are important in determining their spatial structure and ultimate evolution.

Flow

The preceding discussion ignores a special and important non-trivial solution to (1.23): $w = 0$. This solution corresponds to steady solutions to the linear Boussinesq equations, in which case (1.17) through (1.19) reduce to

$$f_0 v = p_x \,, \tag{1.27}$$

$$-f_0 u = p_y \,, \tag{1.28}$$

$$b = p_z \,. \tag{1.29}$$

Equations (1.27) and (1.28) are the conditions of geostrophic balance and (1.29) is the condition of hydrostatic balance. Geostrophic flow obeys $u_x + v_y = 0$ and can be described by the geostrophic streamfunction

$$\psi \stackrel{\text{def}}{=} p / f_0 \,, \qquad \text{so that} \qquad (u, v, b) = (-\psi_y, \psi_x, f_0 \psi_z) \,. \tag{1.30}$$

Unlike internal waves, geostrophic flow does not evolve in the linear Boussinesq equations with $f = f_0$. Its evolution must appeal either to nonlinearity or effects of the Earth's curvature through β.

Limits of linearity. In the nonlinear equations in (1.7) through (1.11), both waves and flow acquire slow but non-infinite time-scales associated with slight departures from the linear balances in (1.17) through (1.21). If we denote the fast wave time-scale $\tilde{t}$ and the flow time-scale $\bar{t}$, the nearly-linear solutions to (1.7) through (1.11) become

$$Q = Q(\boldsymbol{x}, \bar{t})\,, \qquad \text{and} \qquad w(\boldsymbol{x}, \tilde{t}, \bar{t}) = \sum_n \mathrm{e}^{-\mathrm{i}\sigma_n \tilde{t}} A_n(\boldsymbol{x}, \bar{t})\,. \tag{1.31}$$

The methods of this dissertation are, crudely put, to *(i)* derive an equation for the slow evolution of Q which isolates the 'average' effects of waves over the long time-scales of $\bar{t}$, and *(ii)* restrict attention to one or two frequencies σ_n and derive slow evolution equations for A_n that couple to the slow evolution of Q. We next discuss how to isolate the slow evolution of Q from (1.7) through (1.11) in the classic case of quasi-geostrophic flow.

1.2.3 Interaction and non-interaction of waves and flow

One of the main accomplishments of this dissertation is the definition of a new material invariant named 'Available Potential Vorticity', or APV. A comprehensive introduction to APV is given in chapter 2.2. One definition of APV is

$$Q(\boldsymbol{x}, t) \stackrel{\text{def}}{=} \Pi(\boldsymbol{x}, t) - \Pi_*(\boldsymbol{x} - \boldsymbol{\Xi})\,, \tag{1.32}$$

where Π is Ertel PV defined in (1.15), $\Pi_* \stackrel{\text{def}}{=} fN^2$ is its static 'background' part, and $\boldsymbol{\Xi}(\boldsymbol{x}, t)$ is exact nonlinear particle displacement defined through $\mathrm{D}_t\boldsymbol{\Xi} = \boldsymbol{u}$. Because $\mathrm{D}_t\Pi = 0$ and $\mathrm{D}_t\left(\boldsymbol{x} - \boldsymbol{\Xi}\right) = 0$, APV is materially conserved, so that

$$\mathrm{D}_t Q = 0\,. \tag{1.33}$$

APV isolates the part of potential vorticity with a meaningful, intrinsic evolution. When $f = f_0$ is constant, Q expands for $\omega/f_0 \sim b_z/N^2 \ll 1$ into

$$Q = N^2\left[\omega + \partial_z\left(\frac{f_0 b}{N^2}\right)\right] + \boldsymbol{\omega}\cdot\boldsymbol{\nabla} b - \frac{f_0\Lambda_{zz}}{N^2}\tfrac{1}{2}b^2 + \cdots\,, \tag{1.34}$$

where $\Lambda \stackrel{\text{def}}{=} \ln N^2$.

The APV equation opens a relatively straightforward path to the result that the evolution of quasi-geostrophic flow is independent from waves *of equal 'magnitude'* to leading-order in Rossby number. This result was shown by Bartello (1995) and Majda & Embid (1998) for the rotating Boussinesq equations and by Warn (1986) and Dewar & Killworth (1995) for the shallow water equations. We define two non-dimensional parameters,

$$\epsilon \stackrel{\text{def}}{=} \frac{U}{f_0 L}\,, \qquad \text{and} \qquad Bu \stackrel{\text{def}}{=} \left(\frac{N_0 H}{f_0 L}\right)^2\,, \tag{1.35}$$

where N_0, U, H, and L are characteristic scales for N, velocity, height, and horizontal extent of the motion. The parameter ϵ, which is both the Rossby number as well as a measure of wave amplitude, is assumed small. Note that this definition of ϵ differs from that in section 2.3.1, where ϵ is a measure of wave amplitude only and the Rossby number is ϵ^2. The parameter Bu is the Burger number, which measures the magnitude of the horizontal pressure gradient relative to inertia. The ratio f_0/N_0 is almost always small in the Earth's ocean except for isolated, abyssal places. The standard quasi-geostrophic assumption is that $H/L \sim f_0/N_0 \ll 1$ such that $Bu = O(1)$. This assumption reduces the vertical momentum equation (1.9) to the statement of hydrostatic balance, $p_z = b$.

The bread-and-butter asymptotic method of this dissertation is the multiple-scale 'two-time' expansion, which assumes the existence of two time-scales: a fast wave time-scale $\tilde{t} \sim f_0^{-1}$, and a slow flow-evolution time-scale $\bar{t} \sim (\epsilon f_0)^{-1}$. Time-derivatives are accordingly split into

$$\partial_t \mapsto \partial_{\tilde{t}} + \epsilon\, \partial_{\bar{t}}\,, \tag{1.36}$$

The non-dimensional APV equation becomes

$$Q_{\tilde{t}} + \epsilon \left(\boldsymbol{u} \cdot \boldsymbol{\nabla} Q + Q_{\bar{t}} \right) = 0\,. \tag{1.37}$$

All quantities are expanded in ϵ, so that APV has the expansion

$$Q = \underbrace{N^2 \left[\omega_0 + \partial_z \left(\tfrac{b_0}{N^2} \right) \right]}_{\stackrel{\text{def}}{=} Q_0} + \epsilon \Big(\underbrace{\boldsymbol{\omega}_0 \cdot \boldsymbol{\nabla} b_0 + N^2 \left[\omega_1 + \partial_z \left(\tfrac{b_1}{N^2} \right) \right]}_{\stackrel{\text{def}}{=} Q_1} \Big) + \cdots \tag{1.38}$$

Notice that Q_0 is just the linear APV from (1.22).

The leading-order velocity $\boldsymbol{u}_0$ obeys the linear equations (1.17) through (1.21) with hydrostatic balance $p_{0z} = b_0$ replacing (1.19). By averaging over the fast time-scale, $\boldsymbol{u}_0$ can be decomposed into waves, $\tilde{\boldsymbol{u}}_0$, and flow $\bar{\boldsymbol{u}}_0$,

$$\boldsymbol{u}_0 = \bar{\boldsymbol{u}}_0 + \tilde{\boldsymbol{u}}_0\,. \tag{1.39}$$

The average is defined so that $\bar{\tilde{a}} = 0$ and $\overline{(a_{\tilde{t}})} = 0$, when $a(\boldsymbol{x}, \tilde{t}, \bar{t})$ is any variable decomposed into fast and flow components. $\tilde{\boldsymbol{u}}_0$ is a rapidly oscillating wave field governed approximately by (1.23) and $\bar{\boldsymbol{u}}_0$ is slowly-evolving quasi-geostrophic flow. $\bar{\boldsymbol{u}}_0$ obeys geostrophic and hydrostatic balance and can thus be expressed by a geostrophic streamfunction,

$$\psi \stackrel{\text{def}}{=} \bar{p}_0\,, \qquad \text{so that} \qquad \left(\bar{u}_0, \bar{v}_0, \bar{b}_0 \right) = \left(-\psi_y, \psi_x, \psi_z \right)\,. \tag{1.40}$$

The non-interaction result follows in two-steps. At leading-order, the APV equation amounts to a restatement of (1.22),

$$N^{-2} Q_{0\tilde{t}} = \partial_{\tilde{t}} \left[\omega_0 + \partial_z \left(\frac{b_0}{N^2} \right) \right] = 0\,. \tag{1.41}$$

The integral of (1.41) implies that $Q_0 = \bar{Q}_0(\boldsymbol{x}, \tau)$ does not depend on the fast time. The $O(\epsilon)$ terms in the APV equation (1.37) are

$$Q_{0\bar{t}} + Q_{1\tilde{t}} + \boldsymbol{u}_0 \cdot \boldsymbol{\nabla} Q_0 = 0\,. \tag{1.42}$$

Because Q_0 does not depend on the fast time $\tilde{t}$, the time-average of (1.42) is

$$Q_{0\bar{t}} + \bar{\boldsymbol{u}}_0 \cdot \boldsymbol{\nabla} Q_0 = 0\,. \tag{1.43}$$

Equation (1.43) is the ordinary quasi-geostrophic equation. If we restore dimensionality, and define the 'quasi-geostrophic potential vorticity' as $q = Q_0/N^2$, (1.43) rearranges into the 'standard' quasi-geostrophic equation with $\beta = 0$,

$$q_{\bar{t}} + \mathrm{J}\left(\psi, q\right) = 0\,, \qquad \text{with} \qquad q \stackrel{\text{def}}{=} \left(\partial_x^2 + \partial_y^2 + \partial_z \frac{f_0^2}{N^2}\partial_z\right)\psi\,. \tag{1.44}$$

The operator $\mathrm{J}(a,b) = a_x b_y - a_y b_x$ is the Jacobian so that $\partial_t + \mathrm{J}(\psi,\cdot) = \partial_t + \bar{\boldsymbol{u}}_0 \cdot \boldsymbol{\nabla}$ is the wave-averaged and leading-order material derivative. To time-scales at least as long as $(\epsilon f_0)^{-1}$, the evolution of q is independent from $\tilde{\boldsymbol{u}}_0$ and thus internal waves. On longer time-scales, however, the independence of q and $\tilde{\boldsymbol{u}}_0$ is not secure.

1.3 The shape of things to come

This dissertation develops models in which waves and flow *coevolve* and interact with two-way coupling. For this purpose we revise the assumption in chapter 1.2.3 that both waves and flow are leading-order solutions to (1.7) through (1.11). Instead, we assume that waves are 'strong', and flow is 'weak', so that $\boldsymbol{u}_0 = \tilde{\boldsymbol{u}}_0$ and the leading-order solution of (1.7) through (1.11) is a rapidly oscillating wave field. In this case, the quasi-geostrophic flow is part of the first-order velocity $\boldsymbol{u}_1$, the leading contribution to APV is Q_1, the small parameter ϵ measures wave steepness, and the Rossby number is $Ro = \epsilon^2$.

The work of chapter 2 is then to find a slow evolution equation for $q = Q_1/N^2$. This equation resembles the classical quasi-geostrophic equation in (1.44) but for two crucial differences: first, geostrophic balance is modified and obeyed only by the Lagrangian-mean flow, rather than the Eulerian-mean. The modified balance conditions are given in (2.50) and differ from the traditional balance conditions in (1.28) and (1.27). Second, waves contribute to the APV balance that defines q in (1.44). In consequence the APV equation in (2.1) and (2.2) becomes, with $\beta = 0$,

$$q_{\bar{t}} + \mathrm{J}\left(\psi, q\right) = 0\,, \qquad \text{with} \qquad q \stackrel{\text{def}}{=} \left(\partial_x^2 + \partial_y^2 + \partial_z \frac{f_0^2}{N^2}\partial_z\right)\psi + q^{\mathrm{w}}\,. \tag{1.45}$$

Compare (1.45) to (1.44). The new 'wave contribution to APV', q^{w} in (1.45), is defined in (2.3) and modifies the evolution of quasi-geostrophic flow. The surprisingly mundane and kinematic origins of q^{w} are discussed in chapter 2.4.

The contribution of q^{w} to q in (1.45) does *not* imply that 'waves have APV'. The APV in q is still a material invariant advected on the time-averaged particle trajectories described by ψ and decidedly a quantity wholly separate from waves. Instead, the inclusion of q^{w} in the APV balance implies that waves are associated with their own, wave-induced balanced flow that partakes in flow evolution by advecting q. We make this explicit by exploiting the fact that q is linear in ψ, which permits the decomposition

$$\psi = \psi^q + \psi^{\mathrm{w}}\,, \tag{1.46}$$

where ψ^q and ψ^{w} are defined through

$$q = \left(\partial_x^2 + \partial_y^2 + \partial_z \frac{f_0^2}{N^2} \partial_z \right) \psi^q \,, \quad \text{and} \quad - q^{\mathrm{w}} = \left(\partial_x^2 + \partial_y^2 + \partial_z \frac{f_0^2}{N^2} \partial_z \right) \psi^{\mathrm{w}} \,. \tag{1.47}$$

The balanced flow thus has two parts: an ordinary, APV-associated part in ψ^q, and a wave-induced part in ψ^{w}. The effect of waves on flow evolution is expressed entirely in the advection of q by ψ^{w}.

The wave-induced balanced flow ψ^{w} is a nonlinear correction that refines linear hydrostatic wave solutions to better satisfy the nonlinear equations in (1.7) through (1.11). Because infinite plane progressive waves are exact solutions to the nonlinear equations (1.7) through (1.11) when N and f are constant, such waves have $q^{\mathrm{w}} = 0$ and no wave-induced balanced flow. Even vertically-standing but horizontally infinite waves have no wave-induced flow because q^{w} and ψ^{w} are horizontally uniform. In that case ψ^{w} corresponds to steady z-dependent corrections to the pressure and buoyancy fields. Deeper intuitions on wave-induced balanced flows are developed in chapter 2.5.

Infinite plane waves are mathematical figments that do not exist in the Earth's ocean where wave forcing is time-varying and spatially-modulated, f and N are not constant, and heterogeneities like quasi-geostrophic flow advect, refract, and otherwise distort wave fields aspiring to linearity. Such distortion enhances wave field nonlinearity, leading to stronger ψ^{w} and wave 'feedback' on flow evolution and exposing the incompleteness of equation (1.45): the wave property q^{w} is not known in general and worse, depends on ψ^q and q. To close the APV equation in (1.45) we need an equation that describes the slow evolution of the wave field and ψ^{w} in quasi-geostrophic flow describe by the distribution of q. This is the goal of chapters 3 and 4, which separately focus on coupling (1.45) to one of the two conspicuous peaks in figure 1.1: the near-inertial peak in chapter 3, and the tidal peak in chapter 4.

The derivation of the near-inertial equation in chapter 3.3 is particularly tractable due to the weak dispersion of near-inertial waves. Motivated by observations and simulations of the Boussinesq equations that persistently observe near-inertial second harmonic waves with frequency $2f_0$ when near-inertial waves interact with quasi-geostrophic flow (D'Asaro *et al.*, 1995; Niwa & Hibiya, 1999; Danioux *et al.*, 2008), the model is extended to include the nonlinear production and slow evolution of waves frequency $2f_0$. The result is a closed three-component model that describes the simultaneous evolution of APV, the amplitude of the near-inertial waves, and the amplitude of the near-inertial second harmonic. Peculiarly, the two distinct adiabatic invariants of the model identified in chapter 3.6 imply that near-inertial waves can extract energy from quasi-geostrophic flows under ordinary oceanic conditions. Chapter 3.7 compares numerical solutions to the three-component model with the Boussinesq equations and chapter 3.8 discusses the physics these solutions imply.

The interaction between internal tides and quasi-geostrophic flow is tackled in chapter 4. Distilling the slow evolution of internal tides is more difficult than the near-inertial case and equivalent to finding a slow evolution equation for general-frequency hydrostatic internal waves in quasi-geostrophic flow. Key to deriving the internal tide model is the method of reconstitution (Roberts, 1985), which in a sense generalizes the derivation of the $2f_0$ equation in chapter 3. Two solutions to the hydrostatic wave model for barotropic flow are discussed in chapter 4.5. Further work remains to couple the slow hydrostatic wave evolution to the modified quasi-geostrophic system in (2.1) through (2.3).